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ON LATERAL BUCKLING OF A SLENDER CANTILEVER BEAM

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Abstract—The problem of lateral buckling in a slender elastic cantilever beam under the simultaneous action of a uniformly distributed load and an axial force was studied, as well as shape and load imperfections. First, the perfect case was considered, in which imperfections did not exist, and second, the imperfect case where the imperfections were enclosed. The first interaction curve, which corresponds to the perfect case, was obtained. Dunkerley's straight interaction line is also given. The Liapunov–Schmidt method was used to obtain the bifurcation, i.e. the branching equation.

1. INTRODUCTION

The present article is concerned with the solution of the problem of lateral buckling in a slender linearly elastic cantilever beam under the simultaneous action of a uniformly distributed and an axial force. In particular, we consider a narrow prismatic cantilever beam, and the stability problem for the case where shape and load imperfections are involved. In the stability analysis we shall use bifurcation methods (Chow and Hale, 1982) and singularity theory (Golubitsky and Schaeffer, 1985).

The first studies of lateral buckling in a slender elastic beam were done by Michell (1899) and Prandtl (1900). Previous reviews of lateral buckling problems can be found in Hodges and Peters (1975) and Reissner (1979).

In this article a different derivation, related to the previously published equilibrium equations of deformed cantilever beams, is given. The purpose of this paper is to study the lateral buckling problem as a two-parameter bifurcation problem. The two-parameter elastic stability problem has received considerable attention.

2. EQUATIONS OF EQUILIBRIUM

2.1. *Description of the model*

We consider an incompressible elastic slender cantilever beam, of length l. Its cross-section is rectangular or arbitrarily symmetrical. The beam is composed of a homogeneous isotropic material. The shape imperfection is given by the small initial curvature $1/R_0$ of the center line of the beam. We assume, without loss of generality, that the initial geometrical imperfection is a three-dimensional curved torsionless center line. A uniformly distributed load q acting at the center line of the homogeneous cantilever beam, is parallel to the vertical direction. A compressive horizontal force H acting at the centroid T of the end cross-section of a cantilever beam, is parallel to the horizontal direction. The cantilever beam is assumed to be subjected to loads q and H acting simultaneously. The load imperfection is given by a small horizontal end force P, in the lateral direction.

The deflection of the considered beam occurs under the action of the above-described loads, i.e. it will appear as an equilibrating configuration corresponding to the lateral buckled cantilever beam. Its center line becomes a space elastic curve (Popov, 1948), with the arbitrary large curvature and torsion shown in Fig. 1.

We shall assume a Bernoulli–Euler hypothesis for the bending, and a Saint-Venant hypothesis for the twisting of beams. In addition, we assume that the considered cantilever beam is made of a Hookean material, with modulus of elasticity E and shear modulus G. The shear deformations of the beam are neglected. For the cross-section of a slender beam, with principal moments of inertia $I_1 > I_2$ and torsional constant $I_3 = I_t$, the flexural and torsional rigidities are EI_1, EI_2, and GI_t, respectively.

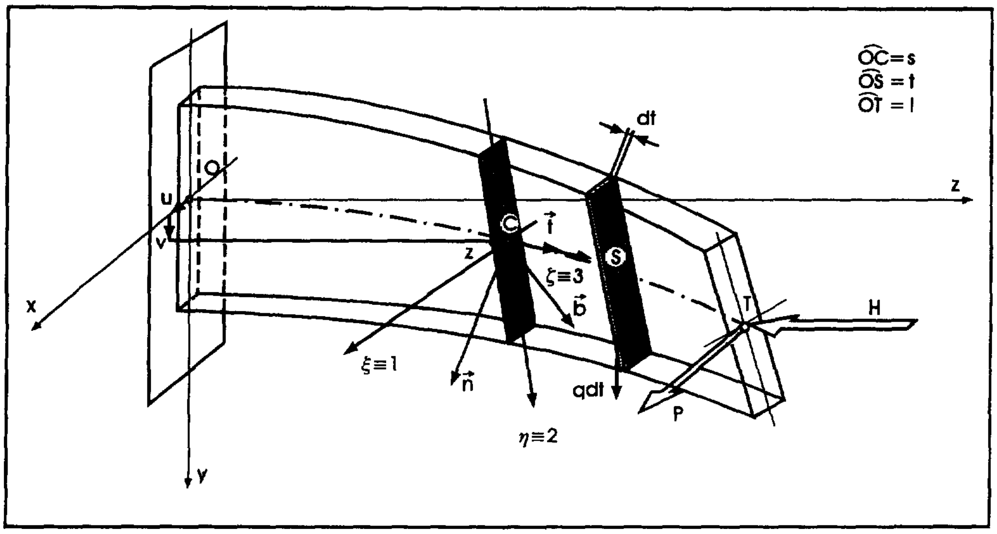


Fig. 1. Loading parameters of the cantilever beam and cantilever geometry.

Let us consider an arbitrarily deflected elastic curve OT $= l$ in space, as shown in Fig. 1. An arbitrary point C on the elastic curve is defined by an increasing arc-length OC $= s$, which is accepted as an independent variable. We will use two cartesian systems of coordinates, a fixed $Oxyz$ and a movable $C\xi\eta\zeta \equiv$ C123. The coordinate system C123 whose origin C moves along the elastic curve is called the principal system of flexure and torsion. The axes ξ and η coincide with the axes of the principal moments of inertia of the cross-section. The ζ-axis is in the direction of the tangent to the elastic curve. The unit vectors along the fixed axes and the moving axes are $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ and $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, respectively. The Frenét system **Ctnb** is attached to an arbitrary deflected elastic line in space in the sense of differential geometry. The unit vectors of the tangent, normal, and binormal directions are $\mathbf{t}$, $\mathbf{n}$, and $\mathbf{b}$. In general, the principal system is not identical with the Frenét system.

When imperfections do not exist ($1/R_0 = 0$, $P = 0$), then we have the unloaded cantilever beam with two planes of symmetry: the (x, z)- and (y, z)-planes. This is *the perfect case* of the considered problem. Otherwise, it is *the imperfect case*.

The position vector of a point C relative to the origin O of the cartesian system $Oxyz$ is $\mathbf{r}(s) = \{u(s), v(s), z(s)\}$. Similarly, the position vectors of points C and T are $\mathbf{r}(t)$ and $\mathbf{r}(l)$, respectively. Here, OS $= t$ and OT $= l$. The tangent, normal and binormal directions of the Frenét system are given by

$$\mathbf{t} = \{u', v', z'\}, \quad \mathbf{n} = R\{u'', v'', z''\}, \quad \mathbf{b} = \mathbf{t}\times\mathbf{n}.$$

where R is the radius of curvature of the space elastic curve at the point C. The primes indicate differentiation with respect to the arc-length s.

2.2. *The external moment*

A cantilever beam is subjected to the uniformly distributed load q along the arc-length, i.e. to the elementary force $\mathrm{d}\mathbf{Q}(s) = \{0, q\mathrm{d}s, 0\}$, $0 \leqslant s \leqslant l$, and to the end force $\mathbf{F} = \{P, O,$

$-H\}$. Let as consider the equilibrium of the part of the cantilever beam to the right of any cross-section with the centroid C. The moment about any point C of the elastic curve is

$$\mathbf{M}_e = \int_s^l (\mathbf{r}(t) - \mathbf{r}(s)) \times \mathrm{d}\mathbf{Q} + (\mathbf{r}(l) - \mathbf{r}(s)) \times \mathbf{F},$$

where ( ) × ( ) denotes the cross product of vectors. The scalar components of the $\mathbf{M}_e$ in the axes of the principal system of flexure and torsion, i.e. $C\xi\eta\zeta \equiv C123$ are $M_1 = \mathbf{M}_e \cdot \mathbf{e}_1$, $M_2 = \mathbf{M}_e \cdot \mathbf{e}_2$, $M_3 = \mathbf{M}_e \cdot \mathbf{e}_3$, where ( ).( ) denotes the dot product of vectors. The external moment is:

$$\mathbf{M}_e = \{M_1, M_2, M_3\}.$$

2.3. *The internal moment*

Under the action of external loads, a cantilever beam will be bent about the binormal direction, and will be twisted about the tangent direction of the Frenét system. The internal moment $\mathbf{M}_i$ about the point C contains three scalar components in the axes of the principal system of flexure and torsion. The scalar components of the internal moment $\mathbf{M}_i$ are proportional to the difference between the actual curvature and the curvature in the tensionless state [see for example, Popov (1948)]. So, it follows that

$$M_{f_1} = EI_1\left(\frac{1}{R}(\mathbf{b}\cdot\mathbf{e}_1) - \frac{1}{R_0}(\mathbf{b}\cdot\mathbf{e}_1)_0\right); \quad \frac{1}{R_0}(\mathbf{b}\cdot\mathbf{e}_1)_0 = \frac{k}{l}k_{01},$$

$$M_{f_2} = EI_1\left(\frac{1}{R}(\mathbf{b}\cdot\mathbf{e}_2) - \frac{1}{R_0}(\mathbf{b}\cdot\mathbf{e}_2)_0\right); \quad \frac{1}{R_0}(\mathbf{b}\cdot\mathbf{e}_2)_0 = \frac{k}{l}k_{02},$$

$$M_t = GI_t\beta'(\mathbf{t}\cdot\mathbf{e}_3),$$

where β is the angle of twist, and $(k/l)k_{01}$, $(k/l)k_{02}$ scalar components of the initial curvature in the principal axes of the cross-section in the tensionless state of the cantilever beam. Now, the internal moment expressed in scalar components in the axes of the principal system is:

$$\mathbf{M}_i = \{M_{f_1}, M_{f_2}, M_t\}.$$

2.4. *Equilibrium*

Consider first the connection between the fixed $Oxyz$ and moving $C\xi\eta\zeta$ systems, i.e. relationship between their unit vectors. It is now convenient to introduce an important type of Eulerian angle. We will use the type that has been widely employed in engineering problems, for example in gyrodynamics and robotics. These angles may be identified as Krylov angles. The order of the Krylov rotation can be visualized by the Cardan suspension of gyroscopes. Since β is the angle of twist and $(\mathrm{d}s)^2 = (\mathrm{d}u)^2 + (\mathrm{d}v)^2 + (\mathrm{d}z)^2$, from Fig. 1 it may easily be shown that the transformation from the fixed system $Oxyz$ to the moving system $O\xi\eta\zeta$ is

$$\mathbf{e}_1 = (\cos\beta)(1-v'^2)^{-1/2}\{z' - u'v'\tan\beta,\ \ (1-v'^2)\tan\beta,\ \ -(u' + v'z'\tan\beta\},$$

$$\mathbf{e}_2 = (\cos\beta)(1-v'^2)^{-1/2}\{-(z'\tan\beta + u'v'),\ \ (1-v'^2)\tan\beta,\ \ u'\tan\beta - v'z'\},$$

$$\mathbf{e}_3 = \{u', v', z'\}.$$

Return now to the equilibrium between the external and internal moments, indicated by expressions $\mathbf{M}_e$ and $\mathbf{M}_i$. It is found that

$$M_{f_1} - M_1 = 0\,, \quad M_{f_2} - M_2 = 0\,, \quad M_t - M_3 = 0$$

are the equilibrium conditions for an arbitrary cross-section of the considered cantilever beam in the principal system C123.

At this point, all lengths are normalized by l, i.e. $s = l\bar{s}$, $u = l\bar{u}, \ldots$, $u' = \mathrm{d}u/\mathrm{d}s = \mathrm{d}\bar{u}/\mathrm{d}\bar{s} = \bar{u}', \ldots, u'' = \mathrm{d}^2u/\mathrm{d}s^2 = (1/l)\mathrm{d}^2\bar{u}/\mathrm{d}\bar{s}^2 = (1/l)\bar{u}'', \ldots$. Substituting into the above equilibrium conditions, after simplification and rearrangement, the following equilibrium equations of the considered cantilever beam are found, i.e.

$$V = 0\,, \quad U = 0\,, \quad B = 0,$$

where

$$\begin{aligned} V = & -v'' + (z'u'' - u'z'')\tan\beta - k\,k_{01}(1 - v'^2)^{1/2}/(\cos\beta) \\ & + q_1\left[(z' - u'v'\tan\beta)\int_s^1 (z(t) - z(s))\,\mathrm{d}t + (u' + v'z'\tan\beta)\int_s^1 (u(t) - u(s))\,\mathrm{d}t\right] \\ & + h_1[(z' - u'v'\tan\beta)(v(1) - v(s)) + (1 - v'^2)(u(1) - u(s))\tan\beta] \\ & - p_1[(1 - v'^2)(z(1) - z(s))\tan\beta + (u' + v'z'\tan\beta)(v(1) - v(s))], \end{aligned} \tag{1}$$

$$\begin{aligned} U = & \; v''\tan\beta + (z'u'' - u'z'') - k\,k_{02}(1 - v'^2)^{1/2}/(\cos\beta) \\ & - q_2\left[(z'\tan\beta - u'v')\int_s^1 (z(t) - z(s))\,\mathrm{d}t + (u'\tan\beta + v'z')\int_s^1 (u(t) - u(s))\,\mathrm{d}t\right] \\ & - h_2[(z'\tan\beta + u'v')(v(1) - v(s)) + (1 - v'^2)(u(1) - u(s))] \\ & - p_2[(1 - v'^2)(z(1) - z(s)) - (v(1) - v(s))(u'\tan\beta - v'z')], \end{aligned} \tag{2}$$

$$\begin{aligned} B = \beta' + q_t\left[u'\int_s^1 z(t) - z(s))\,\mathrm{d}t - z'\int_s^1 (u(t) - u(s))\,\mathrm{d}t\right] + h_t[u'(v(1) - v(s)) - u'(u(1) - u(s))] \\ - p_t[v'(z(1) - z(s)) - z'(v(1) - v(s))]. \end{aligned} \tag{3}$$

Expressions (1)–(3) form the system of non-linear integro-differential equations in the dimensionless form. For the sake of simpler writing, the bars are dropped. In these equations, dimensionless loads are:

$$q_k = ql^3/(EI_k)\,, \quad h_k = Hl^2/(EI_k)\,, \quad p_k = Pl^2/(EI_k)\,, \quad k = 1,2;$$
$$q_t = ql^3/(GI_t)\,, \quad h_t = Hl^2/(GI_t)\,, \quad p_t = Pl^2/(GI_t).$$

2.5. *Boundary conditions*

The boundary conditions in dimensionless form, at the fixed end ($s = 0$), and at the free end ($s = 1$) are, respectively,

$$u(0) = v(0) = z(0) = \beta(0) = u'(0) = v'(0) = 0\,, \quad z'(0) = 1\,; \tag{4}$$

$$\beta'(1) = u''(1) = v''(1) = z''(1) = 0. \tag{5}$$

The equations (1)–(3) and boundary condition (4), (5) define a *non-linear two-parameter eigenvalue problem*.

3. REDUCTION TO A BIFURCATION EQUATION

For the reduction of the equilibrium equations (1)–(3), which are essentially non-linear, to the bifurcation equation, the methods of bifurcation of Chow and Hale (1982) and singularity theory (Golubitsky and Schaeffer, 1985), are employed. The analysis of stability of the cantilever beam being considered rests on the bifurcation equation. The imperfection parameters $(\alpha_1, \alpha_2, \ldots, \alpha_m) = \alpha$ are elements of the set $(k_{01}, k_{02}, p_1, p_2, p_t)$. The number and meaning of $\alpha_1, \alpha_2, \ldots, \alpha_m$ will be seen later, as well as the meaning of the two load parameters $(\lambda_1, \lambda_2) = \lambda$.

3.1. *Abstract form of of the equilibrium equation*

In order to derive the bifurcation equation, we rewrite the eigenvalue problem (1)–(5) in an abstract form. We introduce the following spaces of functions: $X = \{x : x \in C^2([0,1]),$ (4)–(5)$\}$, $Y = \{\eta : \eta \in C([0,1])\}$ and the parameter set $\Lambda = \{(\lambda_1,\lambda_2) : \lambda_1,\lambda_2 \in \mathbb{R}^+\}$, which is an open set in a Banach space. The imperfection parameters α belong to an open set in the neighborhood of zero in $\mathbb{R}^m$. The unknown vector $\mathbf{x} \in \mathbf{X}$ is $\mathbf{x} = [v, u, \beta]^{\mathrm{T}}$, where $[\,,\,,\,]^{\mathrm{T}}$ denotes the column vector, C^k denotes the space of the continuous functions mapping the closed interval [0,1] into real $\mathbb{R}$ and having a continuous derivative up to order k, and $\mathbb{R}^+ \subset \mathbb{R}$ denotes the set of non-negative real numbers.

Now let $M : X \times \Lambda \to Y$ be a smooth mapping with X, Y, Λ Banach spaces. We define the non-linear operator M, dependent upon the parameter α having in view expressions (1)–(3), by

$$M(\alpha, \lambda, x) = [V, U, B]^{\mathrm{T}}. \tag{6}$$

Thus, the non-linear eigenvalue problem (1)–(5) is equivalent to the equation

$$M(\alpha, \lambda, x) = M_\alpha(\lambda, x) = 0. \tag{7}$$

Equation (7) describes the *imperfect case* of the lateral buckling problem being considered, while the *perfect case*, without shape and load imperfections, is described by the equation

$$M(0, \lambda, x) = M_0(\lambda, x) = 0. \tag{8}$$

3.2. *The perfect case*

3.2.1. *The linearized eigenvalue problem.* In this section we apply the methods of bifurcation theory to the elastic stability problem we are considering, since it is well known that the eigenvalues of the linearized problem are bifurcation points of the non-linear problem (von Kármán, 1910). A study of the observed case could be made using the linearization of the non-linear operator $M_0(\lambda, x)$, which is the Fréchet derivative $D_x M_0(\lambda, 0)$ at $(\lambda, x) = (\lambda, 0)$, i.e.

$$D_x M_0(\lambda, 0) y = \begin{bmatrix} -v'' + (1/2) q_1 (1-s)^2 + h_1 (v(1) - v(s)) \\ u'' - (1/2) q_2 (1-s)^2 \beta - h_2 (u(1) - u(s)) \\ \beta' + q_t \left((1/2)(1-s)^2 u' - \displaystyle\int_s^1 (u(1) - u(s))\, \mathrm{d}t \right) \end{bmatrix} = 0, \quad y \in X. \tag{9}$$

For the considered slender beam, the flexural rigidity EI_1 is very much greater than EI_2. The first equation of (9) describes the bending deflection of the cantilever beam in the y, z-plane, when a homogeneous lateral load of dimensionless intensity q_1 acts as a cantilever subjected to a compressive dimensionless force h_1. The second and third equations of (9) describe the lateral buckling deflection of a slender cantilever beam. We then have the following system:

$$u'' - (1/2)q_2(1-s)^2\beta - h_2(u(1)-u(s)) = 0,$$

$$\beta' + q_t\left((1/2)(1-s)^2u' - \int_s^1 (u(1)-u(s))\,\mathrm{d}t\right) = 0, \tag{10}$$

subject to the boundary conditions

$$u(0) = u'(0) = \beta(0) = 0; \quad u''(1) = \beta'(1) = 0. \tag{11}$$

By introducing the change of variables $t = 1-s$ and $w = u(0)-u(\mathrm{t})$, the linear eigenvalue problem (10)–(11) transforms into

$$t^2w^{\mathrm{IV}} - 4tw^{\mathrm{III}} + (Bt^6 + At^2 + 6)w'' - 4Atw' + 6Aw = 0, \tag{12}$$

$$w(0) = ((w''+Aw)/t^2)'(0) = w'(1) = (w''+Aw)(1) = 0 \tag{13}$$

where the load parameters are $\lambda_1 = h_2 = A$, $\lambda_2 = (1/4)q_2q_t = B$. Since no closed form solutions exist, eqn (12) can be solved in the generalized power series

$$w(t) = \sum_{n=0}^{\infty} A_n t^{v+n}. \tag{14}$$

in which the number and values of the constants $A_0, A_1, A_2, \ldots, A_n, \ldots$, ($A_0 \neq 0$) are unknown. Substituting eqn (14) in the eqn (12), we obtain the following *indicial equation*:

$$v(v-1)(v-4)(v-5) = 0,$$

and the *recursion formula*

$$(v+n+2)(v+n+1)(v+n-2)(v+n-3)A_{n+2} + B(v+n-4)(v+n-5)A_{n-4} + A(v+n-2)(v+n-3)A_n = 0.$$

The roots of the indicial equation are $v = 0$, 1, 4, 5. Using the recursion formula we obtain $A_1 = A_3 = A_5 = \ldots = 0$ and

$$A_2 = -\frac{A}{(v+2)(v+1)}A_0.$$

First, we take $v(v-1) = 0$ and $(v-4)(v-5) \neq 0$ and using the recursion formula it follows that $A_4 = 0$ and $A_6 = 0$. Thus the generalized series

$$w = A_0t^v\left(1 - \frac{A}{(v+2)(v+1)}t^2 + \frac{AB}{(v+8)(v+7)(v+4)(v+3)}t^8 - \ldots\right)$$

satisfies eqn (12). With $v = 0$, $v = 1$ and $A_0 = 1$ we obtain two linearly independent solutions, i.e.

$$\vartheta = w_1 = w(v=0) = 1 - \frac{A}{2.1}t^2 + \frac{AB}{8.7.4.3}t^8 - \frac{A^2B}{10.9.8.7.4.3}t^{10} + \ldots, \tag{15}$$

$$\psi = w_2 = w(v=1) = t\left(1 - \frac{A}{3.2}t^2 + \frac{AB}{9.8.5.4}t^8 - \frac{A^2B}{11.10.9.8.5.4}t^{10} + \ldots\right). \tag{16}$$

Second, when $v(v-1) \neq 0$ and $(v-4)(v-5) = 0$, the recursive formula gives

$$A_4 = \frac{A^2}{(v+8)(v+7)(v+4)(v+3)}A_0.$$

Now the series satisfying eqn (12) is

$$w = A_0 t^v \left(1 - \frac{A}{(v+2)(v+1)}t^2 + \frac{A^2}{(v+4)(v+3)(v+2)(v+1)}t^4 - \ldots\right).$$

Substituting the roots $v = 4$ and $v = 5$ and $A_0 = 1$, we find two linearly independent solutions, i.e.

$$\varphi = w_3 = w(v=4) = t^4\left(1 - \frac{A}{6.5}t^2 + \frac{AB}{8.7.6.5}t^4 - \left(\frac{A^3}{10.9.8.6.5} + \frac{2B}{10.9.5}\right)t^6 + \ldots\right) \tag{17}$$

$$\chi = w_4 = w(v=5) = t^5\left(1 - \frac{A}{7.6}t^2 + \frac{AB}{9.8.7.6}t^4 - \left(\frac{A^3}{11.10.9.8.7.6} + \frac{B}{11.7.3}\right)t^6 + \ldots\right). \tag{18}$$

The complete solution of eqn (12) is

$$w = A_0\vartheta(t) + A_1\psi(t) + A_4\varphi(t) + A_5\chi(t),$$

where A_0, A_1, A_4, and A_5 are unknown constants.

Under the boundary conditions (13), the functions (14), defined by eqns (15)–(18), form the characteristic equation for the two-parameter eigenvalue problem (12)–(13) being considered:

$$\Delta(A,B) = \Delta(A,B;t=1) = \left[\psi'(1) + \frac{A^2}{5!}\chi'(1)\right][\varphi''(1) + A\varphi(1)]$$
$$- \varphi'(1)[(\psi''(1) + A\psi(1)) + \frac{A^2}{5!}(\chi''(1) + A\chi(1))] = 0, \tag{19}$$

where $\Delta(A, B)$ is the function expressed in the following power series:

$$\Delta(A,B;t) = \sum_{n=1}^{\infty} \mathscr{A}_{2n}t^{2n} \tag{20}$$

The coefficients $\mathscr{A}_{2n}$ were calculated as a function of the dimensionless load parameters A, B. The series in eqn (20) must be truncated to a number of terms (say, $n = N$), i.e. up to a degree (say, $2N$), chosen so that for the obtained numerical results, convergence to the desired accuracy may be ensured. Thus, we obtain ($2N = 40$)

$$\Delta(A,B;t=1)=4.3\left(1-\frac{A}{2!}+\frac{A^2}{4!}-\left(\frac{A^3}{6!}+\frac{B}{6.5}\right)+\left(\frac{A^4}{8!}+\frac{13AB}{8.7.6.5}\right)\right.$$
$$\left.-\left(\frac{A^5}{10!}+\frac{43A^2B}{10.9.8.7.6.3}\right)+\left(\frac{A^6}{12!}+\frac{127A^3B}{12.11.10.9.8.7.5.3}+\frac{B^2}{12.11.6.5}\right)+\dots\right). \quad (21)$$

Equation (21) determines a family of interaction curves $C_0, C_1, \dots$ in the parameter space $\lambda = (\lambda_1, \lambda_2) = (A, B)$. For a stability analysis, only the first interaction curve is important. In Fig. 2 we have plotted the first interaction curve C_0, which corresponds to the critical state of loads $\lambda_0 = (\lambda_1, \lambda_2) = (A, B)_0$, or much simpler $\lambda_0 = (A, B)$. Table 1 shows a few points that determine the interaction curve C_0. The interaction curve C_0 is the two-dimensional manifold of a critical state of the considered lateral buckled cantilever beam. When $\lambda_1 = B = 0$, we get the well-known classical Euler column buckling problem, i.e. $\lambda_{cr} = A_{cr} = \pi^2/4 = 2.4674$. On the other hand, when $\lambda_2 = A = 0$, we obtain the classical Michell–Prandtl lateral buckling problem, i.e. $\lambda_{cr} = B_{cr} = 41.3048$ (Michell, 1899; Prandtl, 1900).

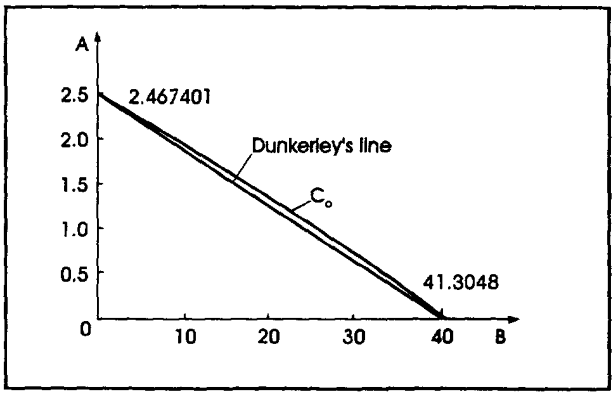


Fig. 2. First interaction curve; Dunkerley's line.

Table 1. Values of A and B on the first interaction curve

$\lambda_{10} = A_{cr} = A$	0	0.5	1.0	1.5	2.0	2.467401
$\lambda_{20} = B_{cr} = B$	41.3048	33.6933	25.7124	17.3516	8.58533	0.00000

3.2.2 *Dunkerley's method.* When the cantilever beam is subjected to different kinds of load parameter simultaneously, then approximate calculation of the critical load parameters by means of Dunkerley's theorem is often used. The interaction curve C_0 representing the simultaneous action of two loading types is shown in Fig. 2. Using Dunkerley's theorem on the considered problem yields Dunkerley's straight interaction line, i.e.

$$\frac{\bar{A}}{A_{cr}}+\frac{\bar{B}}{B_{cr}}=1 \quad \text{or} \quad \frac{\bar{A}}{2.4674}+\frac{\bar{B}}{41.3048}=1.$$

This gives $\bar{A}$, $\bar{B}$, the approximate critical load parameters, and A_{cr}, B_{cr}, the exact critical load parameters, when a single load parameter exists, i.e. if $\bar{B} = 0$ then $\bar{A} = \bar{A}_{cr}$; if $\bar{A} = 0$ then $\bar{B} = \bar{B}_{cr}$. Writing $\bar{A} = A_{cr}\bar{a}$, $\bar{B} = B_{cr}\bar{b}$; $0 \leqslant \bar{a} \leqslant 1, 0 \leqslant \bar{b} \leqslant 1$ reduces the above equation to $\bar{a}+\bar{b} = 1$. Substituting exact values for the critical load, $A = A_{cr}a$, $B = B_{cr}b$, where $A, B \in C_0$, $0 \leqslant a \leqslant 1$, $0 \leqslant b \leqslant 1$ we find from eqn (21) or Table 1 the values for a and b given in Table 2. The error estimate of Dunkerley's straight interaction line may be found as follows.

$$\varepsilon = (a+b)-(\bar{a}+\bar{b}) \equiv (a+b)-1.$$

This is shown in Table 2 as $\varepsilon[\%] = [(a+b)-1] \times 100\%$. The straight interaction line can be use in the design.

Hence $a+b > 1$, which means that for the interaction curve C_0 the well-known *Papkovich's theorem on the convexity of interaction curve* is satisfied.

Table 2. Values of $a = A/A_{cr}$ and $b = B/B_{cr}$ and error estimate $\varepsilon[\%] = [a+b-1] \times 100\%$

a	0	0.20264	0.40528	0.60792	0.81056	1
b	1	0.81572	0.62250	0.41881	0.20785	0
$\varepsilon = (a+b-1) \times 100\%$	0	1.83	2.78	2.67	1.84	0

3.2.3. *The bifurcation equation.* In the study of elastic stability, we consider only the first eigenvector $y_0(\lambda_0)$, which is unique, i.e. $D_x M_0(\lambda_0, 0) y_0 = 0$. By virtue of that, the bounded linear operator $D_x M_0(\lambda_0, 0) = B$ has a one-dimensional null-space $N(B)$ spanned by y_0. Thus, according to Chow and Hale (1982) we can see that dim $N(B) = 1 =$ codim $R(B)$, where R denotes a range in values for the operator B. So, by the Liapunov–Schmidt method we shall obtain one bifurcation equation, for problem (8). Making use of singularity theory, we easily find that $(\lambda, x) = (\lambda_0, 0)$ is a cubic singularity of $M_0(\lambda_0, 0)$, i.e. a bifurcation equation will be a cubic equation. Thus in a sufficiently small neighborhood of $(\lambda_0, 0)$, i.e. $(\lambda, x) = (\lambda_0 + \bar{\lambda}, a y_0 + Z)$, the pair (λ, x) is a solution to eqn (8), and $(\bar{\lambda}, a)$, where† $a \in \mathbb{R}$, satisfies the *bifurcation equation*

$$\bar{F}_0(\lambda, \bar{a}) = \int_0^1 M_0^{\mathrm{T}}(\lambda_0 + \bar{\lambda}, a y_0 + Z) y_0 \, \mathrm{d}s = 0. \tag{22}$$

Since the perturbation of the parameter λ_0 in the neighborhood of $(\lambda, x) = (\lambda_0, 0)$ is denoted by $\lambda = \lambda_0 + \bar{\lambda}$, we may thus write $h_2 = h_{20} + \bar{h}$, $q_2 = q_{20} + \bar{q}$, $q_t = q_{t0} + \bar{q}$. Substituting eqn (8) into eqn (22), at the first eigenmode, we obtain

$$\begin{aligned}
\bar{F}_0(\bar{\lambda}, a) = & -\int_0^1 \Bigg\{ \Bigg[\bar{q} \left(z' \int_s^1 (z(t) - z(s)) \, \mathrm{d}t + a^2 u_0' \int_s^1 (u_0(t) - u_0(s)) \, \mathrm{d}t \right) \tan a\beta_0 \\
& + \bar{h} a (u_0(1) - u_0(s)) \Bigg] u_0(s) + \bar{q} \left(-a u_0'(s) \int_s^1 (z(t) - z(s)) \, \mathrm{d}t \right. \\
& \left. + a z' \int_s^1 (u_0(t) - u_0(s)) \, \mathrm{d}t \right) \beta_0 \Bigg\} \mathrm{d}s + \int_0^1 \Bigg\{ \Bigg[q_{20} \left(\frac{(1-s)^2}{2z'} a\beta_0 \right. \\
& \left. - \left(z' \int_s^1 (z(t) - z(s)) \, \mathrm{d}t + a^2 u_0' \int_s^1 (u_0(t) - u_0(s)) \, \mathrm{d}t \right) \tan a\beta_0 \right) \\
& + h_{20} \left(\frac{1}{z'} - 1 \right) a (u_0(1) - u_0(s)) \Bigg] u_0(s) - \Bigg[q_{t0} \Bigg(a u_0'(s) \left(\frac{1}{2}(1-s)^2 \right. \\
& \left. - \int_s^1 (z(t) - z(s)) \, \mathrm{d}t \right) - (1 - z') a \int_s^1 (u_0(t) - u_0(s)) \, \mathrm{d}t \Bigg) \Bigg] \beta_0 \Bigg\} \mathrm{d}s = 0.
\end{aligned} \tag{23}$$

By using a power series expansion, eqn (23) takes the form

† It is very important to distinguish the load parameter value α, $A = A_{cr}\alpha$ from α, $\alpha \in \mathbb{R}$, the amplitude of the solution to the linearized problem.

$$\bar{F}_0(\bar{\lambda}, a) = -g^*a + \mathbb{A}a^3 + 0(a^5, a^3|\bar{q}|) = 0, \tag{24}$$

where

$$g^* = c_{11}^*\bar{h} + c_{12}^*\bar{q} = g\mathbb{A}, \quad c_{11}^* = \int_0^1 (u_0(1) - u_0)u_0 \, \mathrm{d}s, \quad c_{12}^* = c_{121} + c_{122},$$

$$c_{121} = \frac{1}{2}\int_0^1 (1-s)^2 u_0 \beta_0 \, \mathrm{d}s, \quad c_{122} = \int_0^1 \left(\int_s^1 (u_0(t) - u_0)\, \mathrm{d}t - \tfrac{1}{2}(1-s)^2 u_0'\right)\beta_0 \, \mathrm{d}s,$$

$$\mathbb{A} = c_{31}h_{20} + c_{321}q_{20} - c_{322}q_{10}, \quad c_{31} = \frac{1}{2}\int_0^1 ((u_0(1) - u_0)u_0 u_0'^2 \, \mathrm{d}s,$$

$$c_{321} = \int_0^1 \left(\frac{3}{4}(1-s)^2 u_0'^2 + \int_s^1 \left(\int_s^t u_0'^2 \, \mathrm{d}r\right)\mathrm{d}t - \frac{1}{6}(1-s)^2\beta_0^2 - u_0' \int_s^1 (u_0(t) - u_0)\, \mathrm{d}t\right) u_0 \beta_0 \, \mathrm{d}s,$$

$$c_{322} = \frac{1}{2}\int_0^1 \left(\int_s^1 \left(\int_s^t u_0'^2 \, \mathrm{d}r\right)\mathrm{d}t - u_0' \int_s^1 (u_0(t) - u_0)\, \mathrm{d}t\right) u_0' \beta_0 \, \mathrm{d}s.$$

All upper constants are positive (see Appendix). Note that since $EI_2 \ll EI_1$, it makes sense to speak of the lateral buckling of the considered cantilever beam. Dividing eqn (24) by $\mathbb{A}$ we obtain the bifurcation equation

$$F_0(\bar{\lambda}, a) = -ga + a^3 + 0(a^5, a^3|\bar{q}|) = 0. \tag{25}$$

As $F_0(0, a) = a^3 + \ldots, \dfrac{\partial F_0}{\partial \bar{\lambda}}(0) = 0$, $\operatorname{rank}(\mathrm{d}^2 F_0)(0) = 2$, and $\operatorname{index}(\mathrm{d}^2 F_0)(0) = 1$, this is the bifurcation function $F_0(\bar{\lambda}, a)$ contact equivalent to the following function:

$$G_0(\bar{\lambda}, a) = -ga + a^3 = -(c_{11}\bar{h} + c_{12}\bar{q})a + a^3. \tag{26}$$

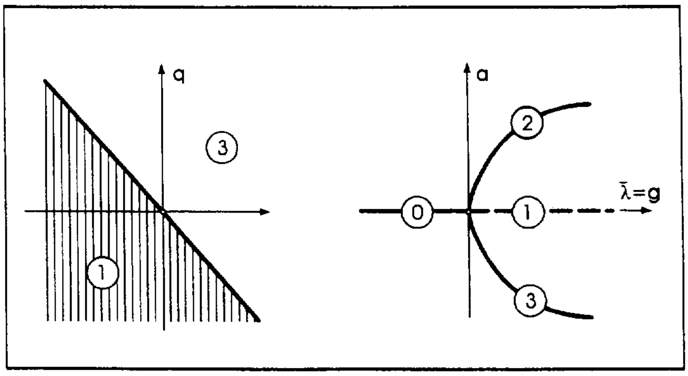
Thus, one can neglect higher-order terms in $F_0(\bar{\lambda}, a)$ without loss of generality in considering the bifurcation problem (Chow and Hale, 1982; Golubitsky and Schaeffer, 1985). Therefore, we may discuss the number of solutions to eqn (8) by considering the equation $G_0(\bar{\lambda}, a) = 0$. Then the number of real solutions to the above equation depends upon the following function:

$$\gamma(\bar{\lambda}) = g = c_{11}\bar{h} + c_{12}\bar{q} \equiv \bar{\lambda}, \tag{27}$$

which is equivalent to the discriminate of a cubic polynomial (26). It follows from eqn (27) that

(i) if $\gamma(\bar{\lambda}) \leqslant 0$: then eqn (8) has exactly one simple solution (λ, x) in the neighborhood of the point $(\lambda, x) = (\lambda_0, 0)$;
(ii) if $\gamma(\bar{\lambda}) < 0$: then eqn (8) has three solutions (λ, x), one simple trivial and one double solution, in the neighborhood of the point $(\lambda, x) = (\lambda_0, 0)$.

The parameter set is given by expression (27), i.e. the number of solutions dependent upon the parameter $\bar{\lambda}$, and represented in Fig. 3(a). *The solution set* of the perfect bifurcation problem being considered is represented in Fig. 3(b). The solid line indicates stable branches and the dashed line indicates the unstable branch. The branch 0 denotes the precritical state, i.e. a stable equilibrium; the branches 1, 2, 3 denote the postcritical state, i.e. branch 1 corresponds to unstable equilibrium, and branches 2 and 3 correspond to a stable equilibrium, i.e. to the laterally buckled state of the cantilever beam being considered.



a) b)

Fig. 3. Perfect bifurcation. (a) The parameter set; (b) the solution set.

3.3. *The imperfect case*

In this section imperfections are introduced into the elastic stability problem under consideration, namely, the shape imperfection and the load imperfection will be given in terms of the initial curvature $1/R_0$ and the small lateral end-force P, respectively. Thus we get a *perturbed*, or *imperfect bifurcation problem* (Golubitsky and Schaeffer, 1985). Therefore, the perturbation of $M(0,\lambda,x) = M_0(\lambda,x)$ controlled by the imperfection parameters $\alpha_1,\alpha_2,\ldots,\alpha_m$ produces a new bifurcation problem

$$M(\alpha,\lambda,x) = 0 \tag{28}$$

at the point $(\alpha,\lambda) = (0,0)$, where $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_m)$. It is easily verified that the operator M has a cubic singularity in the neighborhood of $(\alpha,x) = (0,0)$.

The Liapunov–Schmidt reduction process converts the imperfect bifurcation problem (28), described by expression (6), into the single imperfect bifurcation equation (Chow and Hale, 1982; Golubitsky and Schaeffer, 1985). As in the previous section, we can write

$$\begin{aligned}\bar{F}(\alpha,\bar{\lambda},a) = {} & -\int_0^1 \Big[\frac{kk_{02}}{\cos a\beta_0} + p_2(z(1)-z(s))\Big]u_0(s)\,\mathrm{d}s \\ & -\int_0^1 \Big\{\Big[\bar{q}\Big(z'\int_s^1 (z(t)-z(s))\,\mathrm{d}t + a^2u_0'\int_s^1 (u_0(t)-u_0(s))\,\mathrm{d}t\Big)\tan a\beta_0 \\ & + \bar{h}a(u_0(1)-u_0(s))\Big]u_0(s) + \bar{q}\Big(-au_0'(s)\int_s^1 (z(t)-z(s))\,\mathrm{d}t \\ & + az'\int_s^1 (u_0(t)-u_0(s))\,\mathrm{d}t\Big)\beta_0\Big\}\,\mathrm{d}s + \int_0^1 \Big\{\Big[q_{20}\Big(\frac{(1-s)^2}{2z'}a\beta_0 \\ & -\Big(z'\int_s^1 (z(t)-z(s))\,\mathrm{d}t + a^2u_0'\int_s^1 (u_0(t)-u_0(s))\,\mathrm{d}t\Big)\tan a\beta_0\Big) \\ & + h_{20}\Big(\frac{1}{z'}-1\Big)a(u_0(1)-u_0(s))\Big]u_0(s) - \Big[q_{10}\Big(au_0'(s)\Big(\frac{1}{2}(1-s)^2 \\ & -\int_s^1 (z(t)-z(s))\,\mathrm{d}t\Big) - (1-z')a\int_s^1 (u_0(t)-u_0(s))\,\mathrm{d}t\Big)\Big]\beta_0\Big\}\,\mathrm{d}s = 0.\end{aligned} \tag{29}$$

It is not difficult to obtain the power series expansion of eqn (29), i.e.

$$\bar{F}(\alpha, \bar{\lambda}, a) = -f - ga + ha^2 + a^3 + 0(a^5, a^4(|k_{02}| + p_2), \quad a^3|\bar{q}|), \tag{30}$$

where

$$f = \frac{1}{\mathbb{A}}(c_{01}^* k + c_{02}^* p_2) = c_{01}k + c_{02}p_2, \quad h = \frac{1}{\mathbb{A}}(-c_{21}^* k + c_{22}^* p_2) = -c_{21}k + c_{22}p_2,$$

$$c_{01}^* = \int_0^1 k_{02}u_0 \, \mathrm{d}s, \quad c_{02}^* = \int_0^1 (1-s)u_0 \, \mathrm{d}s,$$

$$c_{21}^* = \frac{1}{2}\int_0^1 k_{02}u_0\beta_0^2 \, \mathrm{d}s, \quad c_{22}^* = \frac{1}{2}\int_0^1 \left(\int_s^1 u_0'^2 \, \mathrm{d}t\right) u_0 \, \mathrm{d}s.$$

The terms g and $\mathbb{A}$ are defined in expression (24).

As an application of singularity theory (Golubitsky and Schaeffer, 1985) to eqn (30), one may introduce the following function:

$$G(\alpha, \bar{\lambda}, a) = -f - ga + ha^2 + a^3. \tag{31}$$

A simple calculation as in the section above, shows that the function $F(\alpha, \bar{\lambda}, a)$ is the contact equivalent to function (31). It is well known that the methods of bifurcation (Chow and Hale, 1982), and singularity theory (Golubitsky and Schaeffer, 1985) imply that the totality of solutions (λ, x) near $(\lambda_0, 0)$ of eqn (7) can be completely determined from the bifurcation equation (30), i.e. from the equation $G(\alpha, \bar{\lambda}, a) = 0$. A further application of singularity theory to the imperfect bifurcation equation (30) now yields *two unique imperfection parameters* $\alpha_1 = -f$ and $\alpha_2 = h$.

Therefore, we see that the imperfect case, which is described by $F(\alpha, \bar{\lambda}, a)$ is the perturbation of the perfect case, i.e. $F(0, \bar{\lambda}, a) = F_0(\bar{\lambda}, a)$, controlled by the imperfection parameters $\alpha = (\alpha_1, \alpha_2) = (-f, h)$. Now we shall prove that $F(\alpha, \bar{\lambda}, a)$ is a universal unfolding of $F_0(\bar{\lambda}, a)$, i.e. that $F(\alpha, \bar{\lambda}, a)$ contains the minimum number of imperfection parameters (Golubitsky and Schaeffer 1985). A simple calculation shows that $J(F) = \det(j(F_{\alpha_1}), j(F_{\alpha_2}), j(F_{\bar{\lambda}}), j(F_a))(0) \neq 0$ and thus $F(\alpha, \bar{\lambda}, a)$ is the universal unfolding of $F_0(\bar{\lambda}, a)$. Here the subscripts denote partial differentiation and j(L) denotes the column vector $(L, L_a, L_{aa}, L_{\bar{\lambda}})^{\mathrm{T}}$. As noted above, the number of real solutions to the equation $G(\alpha, \bar{\lambda}, a) = 0$ depends upon the negative discriminate of the cubic polynomial (31)

$$\gamma = \frac{1}{27}\left[\frac{h^2}{3} + g\right]^3 - \frac{1}{4}\left[\frac{2h^3}{27} + \frac{gh}{3} - f\right]^2. \tag{32}$$

Thus the function (32) gives the exact number of solutions to eqn (7) in the neighborhood of $(\lambda, x) = (\lambda_0, 0)$ for some $\alpha = (\alpha_1, \alpha_2)$ near the origin in $\mathbb{R}^2$. It follows from eqn (32) that

(i) if $\gamma < 0$: then eqn (7) has exactly one simple solution (λ, x) in the neighborhood of the point $(\lambda, x) = (\lambda_0, 0)$;
(ii) if $\gamma = 0$: *then eqn* (7) *has two solutions* (λ, x) in the neighborhood of the point $(\lambda, x) = (\lambda_0, 0)$;
(iii) if $\gamma > 0$: then eqn (7) has three solutions (λ, x) in the neighborhood of the point $(\lambda, x) = (\lambda_0, 0)$.

Now it is not difficult to see that, by using the equation $G(\alpha, \bar{\lambda}, a) = 0$ and function (32), a few different cases of imperfect bifurcations exist. For example, if we keep $0 \leqslant h$,

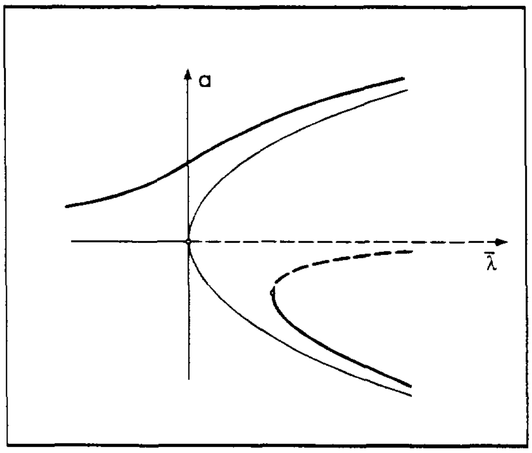


Fig. 4. Imperfect bifurcation—solution set. One-sided supercritical bifurcation.

$0 < f$ then the solution set looks as in Fig. 4. It is a one-sided supercritical bifurcation. There we have indicated the stability assignments using dashed lines for unstable branches.

4. CONCLUSIONS

The objective of this article was to study the problem of lateral buckling of a cantilever beam under the simultaneous action of a uniformly distributed load and an axial force. Using the methods of bifurcation theory and the theory of singularities to the exact equation in an abstract form, we have obtained a bifurcation equation.

(a) *The Perfect Case.* The bifurcation equation (25), i.e. the function (26) corresponds to the nonlinear equation (8). When values of parameters $\lambda = (\lambda_1, \lambda_2)$ occur below the first interaction curve C_0 (see Fig. 2), the considered cantilever has no lateral buckled form, i.e. stable equilibrium paths exist (see the parameter set and the solution set in Figs 3(a) and (b), respectively). When values of the load parameters occur above the curve C_0, the cantilever beam may remain in a laterally buckled configuration (stable equilibrium). Fig. 3(b) shows the solution set, where branches 1, 2 and 3 correspond to the possible postcritical states in a considered cantilever beam.

(b) *The Imperfect Case.* Using the Liapunov–Schmidt method on eqn (7) we have obtained the imperfect bifurcation equation (30), i.e. the function (31). In order to make a more qualitative comparison between the possible imperfect bifurcation, conditions on the coefficients h and f are required. The solution set of the imperfect bifurcation obtained for case when $0 \leqslant h$, $0 < f$ is given in Fig. 4. The initial geometrical imperfection and load imperfection play an important part in modeling real phenomena, particularly when such imperfections can be neglected.

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REFERENCES

Chow, S. N. and Hale, J. K. (1982). *Methods of Bifurcation Theory*. Springer-Verlag, New York.

Golubitsky, M. and Schaeffer, D. G. (1985). *Singularities and Groups in Bifurcation Theory*. Springer-Verlag, New York.

Hodges, D. H. and Peters, D. A. (1975). On the lateral buckling of uniform slender cantilever beams. *Int. J. Solids Structures*. **11**, 1269–1280.

von Kárman, T. (1910). Festigkeitsprobleme in Maschinenbau. In *Encyklopaedie der Matematischen Wissentschaften*, Band IV: *Mechanik* (Edited by F. Klein and C. Mueler), Teilband 4, Heft 3. Teubner, Leipzig.

Michell, A. G. M. (1899). Elastic stability problems of long beams under transverse forces. *Phil. Mag.* (*5th Series*) **48**, 289–389.

Popov, E. P. (1948). *Nonlinear Statics Problems of Thin Rods* (In Russian). GITTL, Moscow.

Prandtl, L. (1900). *Kipp-Erscheinungen, Inaugural-Dissertation* (1899). Buchdrukerei Robert Stich, Nuernberg.

Reissner, E. (1979). On lateral buckling of end-loaded cantilever beams. *ZAMP* **30**, 31–40.

APPENDIX

For a qualitative analysis of the considered lateral buckling problem, some inequalities for the first eigenmode in non-dimensional form are introduced. By using the boundary conditions (4), the auxiliary angle $\theta(s)$, in the form $u' = \sin\theta$, $z' = \cos\theta$, has been introduced.

Proposition 1

$$0 \leqslant z' \int_0^1 (u(t)-u(s))\,\mathrm{d}t - u' \int_0^1 (z(t)-z(s))\,\mathrm{d}t, \quad 0 \leqslant s \leqslant 1. \tag{A.1}$$

Proof: Consider the functions $f(s) = \int_s^1 (z(t)-z(s))\,\mathrm{d}t$ and $g(s) = \int_s^1 (u(t)-u(s))\,\mathrm{d}t$ and the auxiliary angle $\theta(s)$. Applying the Cauchy finite difference theorem to these functions we obtain the inequality (A.1). ■

Proposition 2

$$0 \leqslant z'' \int_0^1 (u(t)-u(s))\,\mathrm{d}t + u'' \int_0^1 (z(t)-z(s))\,\mathrm{d}t, \quad 0 \leqslant s \leqslant 1. \tag{A.2}$$

Proof: Since $0 \leqslant u''$, $0 < z'$ and using inequality (A.1) we obtain the inequality (A.2). ■

Proposition 3

$$0 \leqslant \tfrac{1}{2}(1-s)^2 - \left(z' \int_0^1 (u(t)-u(s))\,\mathrm{d}t + u' \int_0^1 (z(t)-z(s))\,\mathrm{d}t\right), \quad 0 \leqslant s \leqslant 1 \tag{A.3}$$

Proof: By using the Cauchy finite difference theorem for

$$f(s) = z' \int_0^1 (u(t)-u(s))\,\mathrm{d}t - u' \int_0^1 (z(t)-z(s))\,\mathrm{d}t, \quad 0 \leqslant s \leqslant 1,$$

$$g(s) = \tfrac{1}{2}(1-s)^2,$$

and the inequality (A.2), it follows that the inequality (A.3) holds. ■

Proposition 4

$$\tan\beta \leqslant \frac{\beta}{z'}, \quad 0 \leqslant s \leqslant 1. \tag{A.4}$$

Proof: Setting $z' = \cos\theta$, we may introduce the function $F(\beta, \theta) = \tan\beta - \beta/(\cos\theta)$, $0 \leqslant s \leqslant 1$, $0 \leqslant \beta, \theta$. Since the solution of the system $F_\beta = 0$, $F_\theta = 0$ is $(\beta, \theta) = (0, 0)$ and the Hessaian at the point $(0, 0)$ vanishes, a further investigation is necessary. In such a case, we can write $F(h, k)$ for $0 \leqslant h, k$. Without loss of generality, we can put $h = k$. Then, $F(\beta, \theta) - F(0, 0) \leqslant 0$, i.e. F has a maximum at $(0, 0)$. The inequality (A.4) now follows easily from $F(\beta, \theta) \leqslant F(0, 0) = 0$. ■

Proposition 5

$$0 \leqslant \frac{(1-s)^2\beta}{2} - \left(\int_0^1 (z(t)-z(s))\,\mathrm{d}t + u' \int_0^1 (u(t)-u(s))\,\mathrm{d}t\right)\tan\beta, \quad 0 \leqslant s \leqslant 1. \tag{A.5}$$

Proof: Using the inequalities (A.3) and (A.4), we conclude that the inequality (A.5) holds. ■

Proposition 6

$$0 \leqslant u'\left(\tfrac{1}{2}(1-s)^2 - \int_0^1 (z(t)-z(s))\,\mathrm{d}t\right) + (1-z') \int_0^1 (u(t)-u(s))\,\mathrm{d}t, \quad 0 \leqslant s \leqslant 1. \tag{A.6}$$

Proof: Using the Cauchy finite difference theorem for

$$f(s) = \int_0^1 (u(t)-u(s))\,\mathrm{d}t, \quad g(s) = \tfrac{1}{2}(1-s)^2 - \int_0^1 (z(t)-z(s))\,\mathrm{d}t, \quad 0 \leqslant s \leqslant 1,$$

and auxiliary angle $\theta(s)$, it is easy to show that the inequality (A.6) is true. ■

Proposition 7

$$0 \leqslant F(s), \quad 0 \leqslant s \leqslant 1, \tag{A.7}$$

where

$$F(s) = q_{20}\left(\frac{(1-s)^2}{2z'}\beta - \left(z'\int_s^1 (z(t)-z(s))\,\mathrm{d}t + u'\int_s^1 (u(t)-u(s))\,\mathrm{d}t\right)\tan\beta\right)u$$
$$-q_{10}\left(u'(s)\left(\tfrac{1}{2}(1-s)^2 - \int_s^1 (z(t)-z(s))\,\mathrm{d}t\right) - (1-z')\int_s^1 (u(t)-u(s))\,\mathrm{d}t\right)\beta.$$

Proof: The function $F(s)$, which vanishes for $s = 0$ and $s = 1$, is defined and continuous on the closed interval [0, 1], and has a bounded derivative $F'(s)$ on [0, 1] (it may be the open interval (0, 1)). On the basis of the Rolle rule, $F'(s) = 0$, $0 \leqslant s \leqslant 1$. It is easy to show that $F'(0) = F'(1) = 0$, $F''(0) > 0$. Thus we may conclude that inequality (A.7) holds. ■